Tutorial notes — 4 whizkiddeng@gmail.com

Properties of limit:

1. Uniqueness:

once the $\lim_{x\to x_0} f(x_0)$ exists, the value should be unique.

It is useful for us to check a limit doesn't exist as long as we can find 2 different sequences $\{x_k\}$, $\{y_k\}$, $\begin{cases} x_k \to x_0, \\ y_k \to x_0 \end{cases}$, but $\lim f(x_k) \neq \lim f(y_k)$

then we can say $\lim_{x\to x_0} f(x)$ doesn't exist.

i.e. $\sin\frac{1}{x}$ when $x\to 0$.

2. Linearity:

If $\lim_{x\to x_0} f(x)$, $\lim_{x\to x_0} g(x)$ both exist. we have:

① $\lim_{x\to x_0} (f+g) = \lim_{x\to x_0} f + \lim_{x\to x_0} g$

②. $\lim_{x\to x_0} f\cdot g = \lim_{x\to x_0} f \cdot \lim_{x\to x_0} g$

③ $\lim_{x\to x_0} \frac{f}{g} = \frac{\lim f}{\lim_{x\to x_0} g}$ $\left(\lim_{x\to x_0} g(x) \neq 0\right)$

the assumption is important, counter-example like:

①. $f(x)=x$, $g(x)=1-x$ when $x\to+\infty$

$\lim_{x\to+\infty} (f+g) = 1 \neq \lim f(x) + \lim g(x)$

②. $f(x)=x$, $g(x)=\frac{1}{x}$. when $x\to+\infty$

③. $f(x)=x$, $g(x)=x$, $x\to+\infty$

3. Sign-preserving

If $f(x) \le g(x)$ $\Rightarrow$ $\lim_{x\to x_0} f(x) \le \lim_{x\to x_0} g(x)$ the "=" in conclusion is necessary, like:

$(f(x) < g(x))$

in some neighbourhood of x_0.

$\frac{1}{x} < \frac{1}{x-1}$, but when $x\to+\infty$, $\lim_{x\to+\infty} \frac{1}{x} = \lim_{x\to+\infty} \frac{1}{x-1} = 0$.

⇓

Sand-wich thm:

If $h(x) \le f(x) \le g(x)$ in some neighbourhood of x_0 (if x_0 be infinity, that's should hold for x sufficiently large or small)

And $\lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x) = A$

Then $\lim f(x) = A$

Application of Sand-wich Thm.

Q1: If $\lim_{x\to+\infty} f(x) = a$, then try to show $\lim_{x\to+\infty} \frac{[xf(x)]}{x} = a$ where $[\cdot]$ stands for the "floor" function.

Pf: Just recall the definition of floor function. we know:

for $x>0$, we have $x-1 < [x] \leq x$.

so $\frac{xf(x)-1}{x} < \frac{[xf(x)]}{x} \leq \frac{xf(x)}{x} = f(x)$. let $x\to+\infty$, we have: $\lim_{x\to+\infty} \frac{xf(x)-1}{x} = \lim_{x\to+\infty} (f(x) - \frac{1}{x}) = a = \lim_{x\to+\infty} f(x)$

so $\lim_{x\to+\infty} \frac{[xf(x)]}{x} = a$.

Q2: Try to compute $\lim_{x\to+\infty} x^{\frac{1}{x}}$.

Pf: First for the function like $f(x)^{g(x)}$, we often use $x = e^{\ln x}$ to transform it to a simpler case like: $f(x)^{g(x)} = e^{g(x)\ln f(x)}$

Due to the continuity of e^x, we can send the limit inside means:

$\lim_{x\to x_0} f(x)^{g(x)} = \exp\{\lim_{x\to x_0} g(x)\ln f(x)\}$ (such proof needs the rigorous definition of limit which means the ε-δ lanauage)

for $x^{\frac{1}{x}}$, the case reduce to compute $\lim_{x\to+\infty} \frac{\ln x}{x}$

so another trick is "subsititution", use $t = \ln x$, so $e^t = x$

when $x\to+\infty \Rightarrow t\to+\infty$, $\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{t\to+\infty} \frac{t}{e^t}$

recall definition of $e^t = 1 + t + \frac{1}{2!}t^2 + \cdots$ so $e^t > \frac{1}{(k+1)!}t^{k+1}$ for any $k>0$ when t is large.

so $\lim_{t\to+\infty} \frac{t^k}{e^t} \leq \lim_{t\to+\infty} \frac{t^k}{t^{k+1}}(k+1)! = \lim_{t\to+\infty} \frac{(k+1)!}{t} = 0 \Rightarrow \lim_{t\to+\infty} \frac{t^k}{e^t} = 0$

which means e^t is increasing much faster than any polynomials. similarly:

$\ln x$ is increasing slower than any x^{α} as long as $\alpha > 0$.

we may give the order like:

$\ln x < x^{a} < \cdots < x < x^{2} \cdots < e^{x}$ (the "<" just implies the order)

back to $x^{\frac{1}{x}}$, now we try to compute it just by sand-wich thm,

First consider discrete form, $n^{\frac{1}{n}}$ when n is integer.

$n^{\frac{1}{n}} \geq 1$ is clear, then we write $n^{\frac{1}{n}}$ as:

$n^{\frac{1}{n}} = 1 + d_n$. d_n is a number related to n

$\Rightarrow n = (1+d_n)^n = (\text{binomial thm}) = 1 + nd_n + \frac{n(n-1)}{2}d_n^2 + \cdots > \frac{n(n-1)}{2}d_n^2$ (all terms are positive)

$\Rightarrow d_n < \sqrt{\frac{2}{n-1}}$

$\Rightarrow n^{\frac{1}{n}} = 1 + d_n < 1 + \sqrt{\frac{2}{n-1}}$, and $\lim_{n\to+\infty}(1+\sqrt{\frac{2}{n-1}}) = 1$

so by sand-wich thm, $\lim_{n\to+\infty} n^{\frac{1}{n}} = 1$

And for $x^{\frac{1}{x}}$, we can use the derivative to check that $x^{\frac{1}{x}}$ is a strictly decreasing function when x is large.

For any x, there must exist an integer n s.t $n \leq x < n+1$.

from strictly decreasing we know: $(n+1)^{\frac{1}{n+1}} < x^{\frac{1}{x}} \leq n^{\frac{1}{n}}$

(both bounds $\downarrow 1$)

let $x \to +\infty$ which implies $n \to +\infty$, we get $\lim_{x\to+\infty} x^{\frac{1}{x}} = 1$.

Q3. compute $\lim_{x\to 0} \frac{\sqrt[n]{1+x} - 1}{x}$ (n is a integer)

Pf: It is related to the rationalize, means try to get off the ill-part by finding the common factor and cancell it. An useful formula is:

$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$ $(x \neq y)$.

$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + \cdots + y^{n-1})$ ($x \neq y$ and n is odd) in this case, let $t=\sqrt[n]{1+x}$:

$$\sqrt[n]{1+x}-1 = \frac{(\sqrt[n]{1+x})^n - 1^n}{(t^{n-1}+t^{n-2}+\cdots+1)} = \frac{x}{1+t+\cdots+t^{n-1}}$$

so $$\lim_{x\to 0}\frac{\sqrt[n]{1+x}-1}{x} = \lim_{x\to 0}\frac{1}{1+t+\cdots+t^{n-1}} \text{ (when } x\to 0,\ t\to 1) = \frac{1}{n}.$$

Q4: $$\lim_{y\to+\infty}\frac{\sqrt{1+y^3}}{\sqrt{y^2+y^3}+y}$$

Pf: $$\lim_{y\to+\infty}\frac{\sqrt{1+y^3}}{\sqrt{y^2+y^3}+y} = $$ ~~$\lim_{y\to+\infty}\frac{\sqrt{1+y^3}\,(\sqrt{y^2+y^3}-y)}{y^3}$ (rational factor)~~

$$= \lim_{y\to+\infty}\frac{\sqrt{1+y^3}/\sqrt{y^3}}{(\sqrt{y^2+y^3}+y)/\sqrt{y^3}} = \lim_{y\to+\infty}\frac{\sqrt{1+\frac{1}{y^3}}}{\sqrt{1+\frac{1}{y}}+\frac{1}{\sqrt{y}}} = 1.$$

Remark: $$\frac{\sqrt{x^2+1}}{x} = \begin{cases}\sqrt{1+\frac{1}{x^2}}, & x>0\\ -\sqrt{1+\frac{1}{x^2}}, & x<0.\end{cases}$$

it should be careful when put sth into the square root, notice the sign!

Q5. If $\lim_{x\to+\infty}(a\sin x + b\cos x)$ exists, then a, b must be 0.

Pf: this is related to the uniqueness of limit. just as before. $f(x) = a\sin x + b\cos x$.

we choose different sequence like:

$$\begin{cases} x_k = 2k\pi + \frac{\pi}{2}. & k=1,2\cdots \\ y_n = 2n\pi - \frac{\pi}{2}. & n=1,2\cdots \end{cases}$$

then we know $\begin{cases}\lim_{k\to+\infty} f(x_k) = a \\ \lim_{n\to+\infty} f(y_n) = -a\end{cases}$ From the uniqueness, we must have: $a=-a \Rightarrow a=0$

it's similar to prove $b=0$.

* Sandwich thm:

If $f(x) \le g(x) \le h(x)$ $\forall x \in I$ (interval)

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

then $\lim_{x\to a} g(x) = L$

Eg: Show $\lim_{x\to 0} |x| \sin(\frac{1}{x})$

Since $-1 \le \sin(\frac{1}{x}) \le 1$, then $-|x| \le |x| \sin\frac{1}{x} \le |x|$

$\lim_{x\to 0} -|x| = 0$, $\lim_{x\to 0} |x| = 0$

then $\lim_{x\to 0} |x| \sin(\frac{1}{x}) = 0$

* Prove $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

Recall: by definition $e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots$

So $\frac{e^h-1}{h} = \frac{h + \frac{h^2}{2!} + \cdots}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots$

$$\left| \frac{e^h-1}{h} - 1 \right| = \left| \frac{h}{2!} + \frac{h^2}{3!} + \cdots \right|$$
$$\le \left|\frac{h}{2!}\right| + \left|\frac{h^2}{3!}\right| + \cdots$$

Note k-th term is $h^{k-1}/k!$

$k! = 1\cdot 2 \cdot \cdots \cdot k \ge 2^{k-1}$

So $h^{k-1}/k! \le (\frac{h}{2})^{k-1}$

So $$\left|\frac{e^h-1}{h} - 1\right| \le \left|\frac{h}{2!}\right| + \cdots$$
$$\le \left|\frac{h}{2}\right| + \left|\frac{h}{2}\right|^2 + \cdots$$
$$= \left|\frac{h}{2}\right| \cdot \frac{1}{1 - \left|\frac{h}{2}\right|}$$

So $-\left|\frac{h}{2}\right| \frac{1}{1-\left|\frac{h}{2}\right|} \le \frac{e^h-1}{h} - 1 \le \left|\frac{h}{2}\right| \frac{1}{1-\left|\frac{h}{2}\right|}$

But $\lim_{h\to 0} -\left|\frac{h}{2}\right| \frac{1}{1-\left|\frac{h}{2}\right|} = \lim_{h\to 0} \left|\frac{h}{2}\right| \frac{1}{1-\left|\frac{h}{2}\right|} = 0$

So by Sandwich thm,

$\lim_{h\to 0} (\frac{e^h-1}{h} - 1) = 0$, i.e. $\lim_{h\to 0} \frac{e^h-1}{h} = 1$

* **derivative of $f(x)=e^x$**

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=\lim_{h\to 0}e^x\cdot\frac{e^h-1}{h}$$

$$=e^x\lim_{h\to 0}\frac{e^h-1}{h}=e^x$$

* **Determine differentiability of a function at some pt.**

f differentiable at $x=a$

$\Longleftrightarrow$ $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists. $\Longleftrightarrow$ $\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h}$

[useful when cannot compute $\lim_{h\to 0}$ directly]

Example: (Final 2009-2010) Find a, b s.t. f is differentiable at 0

$$f(x)=\begin{cases} x+\frac{\sin x}{x}, & x>0 \\ ax+b, & x\leq 0\end{cases}$$

① f has to be continuous at $x=0$

i.e. $\lim_{x\to 0}f(x)=f(0)=b$

which means $\lim_{x\to 0^+}f(x)=\lim_{x\to 0^-}f(x)=b$

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}\left(x+\frac{\sin x}{x}\right)$$

$$=1 \quad \text{since } \lim_{x\to 0}\frac{\sin x}{x}=1$$

So $b=1$

(2) f is differentiable at $x=0$ iff

$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ exist

$\Leftrightarrow \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$

$$\lim_{h\to 0^+} \frac{h+\frac{\sin h}{h}-1}{h} = \lim_{h\to 0^+} \left(1+\frac{\sin h - h}{h^2}\right)$$

$$= 1 \quad \text{(Exercise 1)}$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{ah+1-1}{h} = a.$$

$\Rightarrow a=1.$

Exercise:

(1) Show $\lim_{h\to 0} \frac{\sin h - h}{h^2} = 0$ using Sandwich thm

(2) Determine whether $f(x) = x|x|$ is differentiable at $x=0$.

Sol'n:

(1) $\sin h = h - \frac{h^3}{3!} + \frac{h^5}{5!} - \cdots$

$$\frac{\sin h - h}{h^2} = \frac{-\frac{h^3}{3!}+\frac{h^5}{5!}-\cdots}{h^2} = -\frac{h}{3!}+\frac{h^3}{5!}-\cdots$$

So $\left|\frac{\sin h - h}{h^2}\right| \le \left|\frac{h}{3!}\right| + \left|\frac{h^3}{5!}\right| + \cdots$

k-th term is $\left|\frac{h^{2k-1}}{(2k+1)!}\right| \leq \left|\frac{h^{2k-1}}{2^{2k}}\right| = \frac{1}{2}\left|\frac{h}{2}\right|^{2k-1}$

So $\left|\frac{\sin h - h}{h^2}\right| \leq \frac{1}{2}\left(\left|\frac{h}{2}\right| + \left|\frac{h}{2}\right|^3 + \left|\frac{h}{2}\right|^5 + \cdots\right)$

$= \frac{1}{2}\left|\frac{h}{2}\right| \cdot \frac{1}{1-\left|\frac{h}{2}\right|^2}$

So $-\frac{1}{2}\left|\frac{h}{2}\right|\frac{1}{1-\left|\frac{h}{2}\right|^2} \leq \frac{\sin h - h}{h^2} \leq \frac{1}{2}\left|\frac{h}{2}\right|\frac{1}{1-\left|\frac{h}{2}\right|^2}$

Since $\lim_{h\to 0} \frac{1}{2}\left|\frac{h}{2}\right|\frac{1}{1-\left|\frac{h}{2}\right|^2} = \lim_{h\to 0} -\frac{1}{2}\left|\frac{h}{2}\right|\frac{1}{1-\left|\frac{h}{2}\right|^2} = 0$

We conclude by sandwich thm that

$$\lim_{h\to 0} \frac{\sin h - h}{h^2} = 0$$

(2) $f(x) = x|x|$ is differentiable at $x=0$ if

$\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ exists.

$f(0)=0$, $f(h) = h|h|$.

$\lim_{h\to 0} \frac{h|h|}{h} = \lim_{h\to 0} |h| = 0$ exists.

So $f(x) = x|x|$ is differentiable at $x=0$.

Tutorial 4

Topics: Computing limits & differentials.

Q1 Compute the following limits

a) $\lim_{x \to \infty} \sqrt{4x^2 + 5x} - 2x$

b) $\lim_{x \to \infty} \dfrac{\sqrt{9x^4 - 3x + 2}}{x^2 - 2x + 5}$

c) $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$

d) $\lim_{x \to 0} \dfrac{\sin(x)}{x}$

Q2 Compute the first derivative of $f(x)$ by using definition.

a) $f(x) = x^{-1/2}$, for $x > 0$

b) $f(x) = \ln(x)$, for $x > 0$

Recall:

- Squeeze theorem: Suppose $f, g, h : \mathbb{R} \to \mathbb{R}$

  If $f(x) \leq g(x) \leq h(x) \quad \forall x \in \mathbb{R}$

  and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = A$

  Then $\lim_{x \to a} g(x) = A$.

- Definition of first derivative.

  Suppose $f : \mathbb{R} \to \mathbb{R}$

  then $f'(x) \overset{\text{def}}{=\!=} \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$\underline{\text{Sol}^n}$

1a) $$\lim_{x \to \infty} \sqrt{4x^2+5x} - 2x$$

$$= \lim_{x \to \infty} \frac{\left(\sqrt{4x^2+5x} - 2x\right)\left(\sqrt{4x^2+5x} + 2x\right)}{\sqrt{4x^2+5x} + 2x}$$

$$= \lim_{x \to \infty} \frac{4x^2 + 5x - 4x^2}{\sqrt{4x^2+5x} + 2x}$$

$$= \lim_{x \to \infty} \frac{5x}{\sqrt{4x^2+5x} + 2x}$$

$$= \lim_{x \to \infty} \frac{5}{\sqrt{4+\frac{5}{x}} + 2} = \frac{5}{4}$$

1b)

$$\lim_{x \to \infty} \frac{\sqrt{9x^4 - 3x + 2}}{x^2 - 2x + 5}$$

$$= \lim_{x \to \infty} \frac{\sqrt{9 - \frac{3}{x^3} + \frac{2}{x^4}}}{1 - \frac{2}{x} + \frac{5}{x^2}}$$

$$= \sqrt{9} = 3$$

1(c) note that

$$-1 \le \cos(y) \le 1 \quad \forall \; y \in \mathbb{R}$$

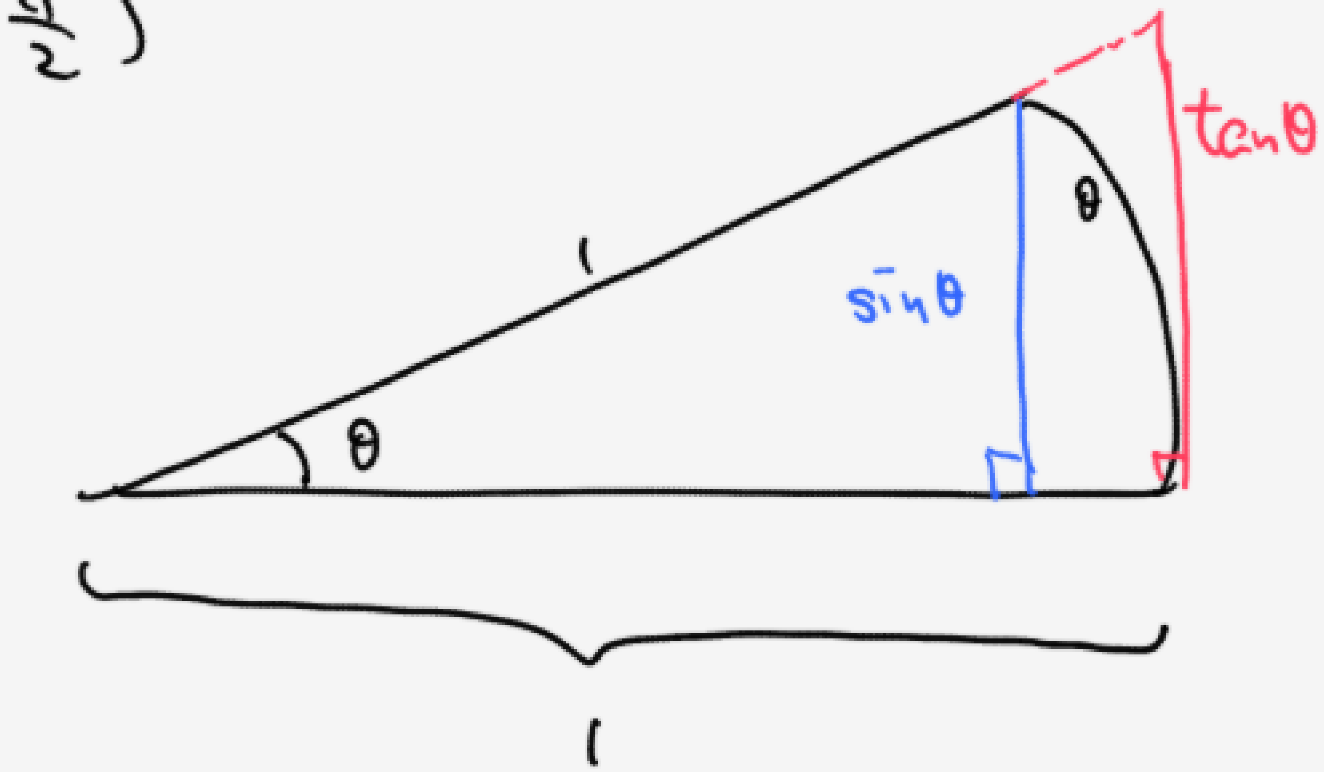
$$\Rightarrow \quad -1 \le \cos\left(\frac{1}{x}\right) \le 1 \quad \forall \; x \in \mathbb{R}$$

$$\Rightarrow \quad -x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2$$

Since $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} (x^2) = 0$

Hence $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$ (by squeeze thm)

(d) by the figure, for any $\theta \in (0, \frac{\pi}{2})$



- $\sin\theta \leq \theta$

  $\Rightarrow \quad \frac{\sin\theta}{\theta} \leq 1$

- $\theta \leq \tan\theta = \frac{\sin\theta}{\cos\theta}$

  $\Rightarrow \quad \cos\theta \leq \frac{\sin\theta}{\theta}$

- hence $\cos x \leq \frac{\sin x}{x} \leq 1 \quad \forall x \in (0, \pi/2)$

  since $\lim_{x\to 0} \cos x = \lim_{x \to 0} 1 = 1 \Rightarrow \lim_{x\to 0} \frac{\sin x}{x} = 1$

  (by squeeze thm)

2a) Given $f(x) = x^{-\frac{1}{2}}$ for $x>0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \cdot \left(\frac{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}} \right)$$

$$= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)}$$

$$= \lim_{h\to 0} \frac{\frac{h}{(x+h)(x)}}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)} = \frac{\frac{1}{x^2}}{\frac{2}{\sqrt{x}}} = \frac{1}{2} x^{-\frac{3}{2}}$$

//

2b) Given $f(x) = \ln x$, for $x > 0$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \ln\left(1 + \frac{h}{x}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\left(\frac{h}{x}\right) - \frac{\left(\frac{h}{x}\right)^2}{2} + \frac{\left(\frac{h}{x}\right)^3}{3} - \cdots \right)$$

$$= \frac{1}{x}$$ //